

# LIMIT THEOREMS FOR COUNTING LARGE CONTINUED FRACTION DIGITS

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**ABSTRACT.** Inspired by a result of Galambos on Lüroth expansions we give a refinement of the famous Borel-Bernstein Theorem for continued fractions and – closely related to this – a Central Limit Theorem for counting large continued fraction digits. As a side result we determine the first  $\phi$ -mixing coefficient  $\phi(1)$  for the Gauss system.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

We establish some 0-1 laws and a central limit theorem for the entries of continued fractions analogous to the work of Galambos in [Gal72], who considered the independent case of entries of the classical Lüroth series.

Throughout the paper, for any irrational number  $x \in \mathbb{R} \setminus \mathbb{Q}$  we will denote its unique infinite regular continued fraction expansion by  $[a_0(x); a_1(x), a_2(x), \dots]$  where

$$x := a_0(x) + \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{\ddots}}} =: [a_0(x); a_1(x), a_2(x), \dots].$$

We may also express this algorithm restricted to  $I := [0, 1)$  by the Gauss map  $G : I \rightarrow I$ ,

$$G(x) := \begin{cases} 1/x - \lfloor 1/x \rfloor, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

With  $G^0 := \text{id}$  and  $G^n := G \circ G^{n-1}$ ,  $n \geq 1$  we obtain the sequence of digits  $a_n(x) := \lfloor 1/G^{n-1}(x) \rfloor$ ,  $n \in \mathbb{N}$ . The algorithm will terminate in  $n \in \mathbb{N}$  only for rational numbers  $x$ , whenever  $G^n(x) = 0$  for the first time; in this way we obtain the finite continued fraction expansion of  $x \in \mathbb{Q}$ .

The transformation  $G$  does not preserve the Lebesgue measure restricted to  $[0, 1]$  denoted by  $\lambda$  (cf. [DK02, Chapter 1.3.3]). However, Gauss found a  $G$ -invariant measure  $\mathbf{m}$  which is equivalent to  $\lambda$  with density  $h(x) = 1/((x+1)\log 2)$ ,  $x \in [0, 1]$  (cf. [IK09, Chapter 1.2.2]). The dynamical system  $([0, 1], \mathcal{B}, G, \mathbf{m})$  is in fact ergodic. Hence by the ergodic theorem, we have for any subset  $A$  of the natural numbers that Lebesgue almost everywhere

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{card} \{k \leq n : a_k \in A\} = \frac{1}{\log 2} \sum_{a \in A} \log \left( 1 + \frac{1}{a(a+2)} \right).$$

Here we used the fact that for all  $z > 0$  we have that

$$\mathbf{m}(a_n \geq z) = \frac{1}{\log 2} \int_0^{1/\lceil z \rceil} \frac{1}{1+x} dx = \frac{1}{\log 2} \cdot \log \left( 1 + \frac{1}{\lceil z \rceil} \right) \quad (1)$$

and

$$\mathbf{m}(a_n > z) = \frac{1}{\log 2} \int_0^{1/(\lfloor z \rfloor + 1)} \frac{1}{1+x} dx = \frac{1}{\log 2} \cdot \log \left( 1 + \frac{1}{\lfloor z \rfloor + 1} \right). \quad (2)$$

*Date:* April 25, 2016.

*2010 Mathematics Subject Classification.* Primary: 11K50 Secondary: 60F20, 60F05.

*Key words and phrases.* continued fractions, 0-1 laws, central limit theorem,  $\phi$ -mixing.

This research was supported by the German Research Foundation (DFG) grant *Renewal Theory and Statistics of Rare Events in Infinite Ergodic Theory* (Geschäftszeichen KE 1440/2-1). TS was supported by the Studienstiftung des Deutschen Volkes.

In particular, every fixed digit  $k \in \mathbb{N}$  will be realized for almost every continued fraction expansion infinitely often. This is no longer true for an increasing sequence  $k_n$  of natural numbers such that  $a_n \geq k_n$  infinitely often. In this paper we give a refinement of the following classical result due to Borel and Bernstein [Bor09, Ber12b, Ber12a].

**Theorem 1.1** (Borel-Bernstein Theorem). *Consider a sequence of positive reals  $(b_n)$ . Then  $a_n \geq b_n$  holds infinitely often with Lebesgue measure 0 or 1, according as the series  $\sum_{n \in \mathbb{N}} 1/b_n$  converges or diverges.*

In fact we are going to prove the following theorem which has partly been considered for independent random variables in the context of Lüroth expansions by Galambos in [Gal72] giving us deeper insights into the growth property of the sequence of digits  $(a_n)_{n \in \mathbb{N}}$ .

**Theorem 1.2.** *Let  $(c_n)_{n \in \mathbb{N}}$  be a sequence of positive real numbers and  $(d_n)_{n \in \mathbb{N}}$  be a sequences of positive integers both tending to infinity. Then*

$$d_n \leq a_n \leq d_n \cdot \left(1 + \frac{1}{c_n}\right)$$

*holds infinitely often with Lebesgue measure 0 or 1, according as*

$$\max \left\{ \sum_{n \in \mathbb{N}} \frac{1}{c_n d_n}, \sum_{n \in \mathbb{N}} \frac{1}{d_n^2} \right\}$$

*is finite or not.*

As a consequence of the above theorem – choosing e.g.  $c_n := 2d_n$  – we get the following corollary.

**Corollary 1.3.** *Let  $(d_n)_{n \in \mathbb{N}}$  be a sequence of positive integers tending to infinity. Then*

$$a_n = d_n$$

*holds infinitely often with Lebesgue measure 0 or 1, according as*

$$\sum_{n \in \mathbb{N}} \frac{1}{d_n^2}$$

*is finite or not.*

**Remark 1.4** For  $d_n := \lfloor \sqrt{n \log(n)} \rfloor$  there are almost surely infinitely many values of  $n$  such that  $a_n = d_n$  and for  $e_n := \lfloor \sqrt{n} \log(n) \rfloor$  there are almost surely only finitely many values of  $n$  such that  $a_n = e_n$ .

Next we state a slightly different version of Theorem 1.2.

**Theorem 1.5.** *Let  $(c_n)_{n \in \mathbb{N}}$  be a sequence of positive real numbers and  $(d_n)_{n \in \mathbb{N}}$  be a sequences of positive integers both tending to infinity. Then*

$$d_n < a_n \leq d_n \cdot \left(1 + \frac{1}{c_n}\right)$$

*holds infinitely often with Lebesgue measure 0 or 1, according as*

$$\sum_{n: c_n \leq d_n} \frac{1}{c_n d_n}$$

*is finite or not.*

As a second main result we give conditions for a central limit theorem (CLT) for a continued fraction counting process to hold. As a corollary ((A) in Corollary 1.9) we obtain a particular CLT connected to the Borel-Bernstein Theorem 1.1 generalizing the central limit theorem stated for the Lüroth coding in [Gal72, Theorem 1]. In Corollary 1.9 we obtain further CLTs connected to Theorem 1.2, Corollary 1.3, and Theorem 1.5.

**Theorem 1.6.** *Let  $(A_n)$  be a sequence of events such that  $A_n \in \sigma(a_n)$  for all  $n \in \mathbb{N}$  and define*

$$\rho := 1 - \frac{1 - \log 2 + \log \log 2}{\log 2} - \frac{2((\pi^2 \log 2)/6 - 1)}{1 - \theta} > 0.68344$$

with  $\theta$  defined in Lemma 2.2. Let  $(A_n)$  be such that there exists  $\epsilon > 0$  such that for only finitely many  $n \in \mathbb{N}$ ,

$$\rho - \epsilon < \mathfrak{m}(A_n) < 1. \quad (3)$$

Let

$$B_n := \begin{cases} A_n & \text{if } \mathfrak{m}(A_n) < 1 \\ \emptyset & \text{else} \end{cases}$$

and suppose

$$\mathfrak{m}\left(\limsup_{n \rightarrow \infty} B_n\right) = 1. \quad (4)$$

Then for  $S_n := \sum_{k=1}^n \mathbb{1}_{A_k}$  we have

$$\lim_{n \rightarrow \infty} \mathfrak{m}\left(\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathbb{V}(S_n)}} < z\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-t^2/2} dt. \quad (5)$$

**Remark 1.7** We have by (2) that

$$\mathfrak{m}(a_n > 1) = \frac{1}{\log 2} \cdot \log\left(1 + \frac{1}{2}\right) < \rho.$$

Thus, if  $(A_n)$  is such that  $B_n \subset \{a_n > 1\}$  for all sufficiently large  $n$ , then the assumption (3) of the theorem is automatically fulfilled.

**Remark 1.8** In order to compare the assumptions (3) and (4) in Theorem 1.6 under the extra condition of  $(X_i := \mathbb{1}_{A_i})$  being a sequence of independent random variables (like for the L uroth system), we make use of Lindeberg's condition to provide necessary conditions for the CLT to hold. That is we assume that for all  $\epsilon > 0$  we have

$$\lim_{n \rightarrow \infty} \frac{1}{\mathbb{V}(S_n)} \cdot \sum_{i=1}^n \mathbb{E}\left((X_i - \mathbb{E}(X_i))^2 \cdot \mathbb{1}_{\{|X_i - \mathbb{E}(X_i)| > \epsilon \cdot \sqrt{\mathbb{V}(S_n)}\}}\right) = 0. \quad (6)$$

We find that this condition is in fact equivalent to

$$\lim_{n \rightarrow \infty} \mathbb{V}(S_n) = \infty \quad (7)$$

by noting that on the one hand condition (7) implies  $\{|X_i - \mathbb{E}(X_i)| > \epsilon \cdot \sqrt{\mathbb{V}(S_n)}\} = \emptyset$  for  $n$  sufficiently large and (6) clearly holds. On the other hand, if  $\lim_{n \rightarrow \infty} \mathbb{V}(S_n) < \infty$ , then there exists  $\epsilon > 0$  such that  $\mathbb{E}\left((X_i - \mathbb{E}(X_i))^2 \cdot \mathbb{1}_{\{|X_i - \mathbb{E}(X_i)| > \epsilon \cdot \sqrt{\mathbb{V}(S_n)}\}}\right) > 0$  for some  $i \in \mathbb{N}$  and consequently (6) fails to hold.

Furthermore, we have that

$$\mathbb{V}(S_n) = \sum_{i=1}^n \mathbb{V}(\mathbb{1}_{A_i}) = \sum_{i=1}^n \mathbb{P}(A_i) \cdot \mathbb{P}(A_i^c),$$

where  $A^c$  denotes the complement of the set  $A$ . This shows that the conditions (3) and (4) stated in the theorem imply condition (7) under the assumption of independence by observing that

$$\sum_{i=1}^n \mathbb{P}(A_i) \cdot \mathbb{P}(A_i^c) = \sum_{i=1}^n \mathbb{P}(B_i) \cdot \mathbb{P}(B_i^c).$$

However, the proof of the Gauss case needs some extra attention due to the lack of independence. In particular, we will provide the exact value of the first  $\phi$ -mixing coefficient for the Gauss system improving an old result of Philipp [Phi88] who obtained 0.4 as an upper bound.

Combing Theorem 1.6 with the 0-1 laws stated above we obtain the following corollary.

**Corollary 1.9.** *Let  $(b_n)$  and  $(c_n)_{n \in \mathbb{N}}$  be arbitrarily chosen sequences of positive real numbers and  $(d_n)_{n \in \mathbb{N}}$  be a sequences of positive integers such that  $(c_n)$  and  $(d_n)$  tend to infinity. Suppose that either*

(A)  $A_n := \{a_n \geq b_n\}$  with  $\sum_{n: b_n > 1} 1/b_n = \infty$ ,

(B)  $A_n := \{a_n = d_n\}$  with  $\sum_{n \in \mathbb{N}} 1/d_n^2 = \infty$ ,

- (C)  $A_n := \left\{ d_n \leq a_n \leq d_n \cdot \left(1 + \frac{1}{c_n}\right) \right\}$  with  $\sum_{n \in \mathbb{N}} 1/(c_n d_n) = \infty$   
or  $\sum_{n: d_n > 1} 1/d_n^2 = \infty$ ,
- (D)  $A_n := \left\{ d_n < a_n \leq d_n \cdot \left(1 + \frac{1}{c_n}\right) \right\}$  with  $\sum_{n: c_n \leq d_n} 1/(c_n d_n) = \infty$ ,
- then for  $S_n := \sum_{k=1}^n \mathbb{1}_{A_k}$  the CLT in (5) holds.

**1.1. Khinchine's Theorem and related results.** In this section we are going to state analogous results to the famous Khinchine 0-1-law for Diophantine approximation which can be stated as follows [Khi35].

**Theorem 1.10** (Khinchine's Theorem). *Let  $k : \mathbb{N} \rightarrow (0, \infty)$  be non-increasing. Then we have*

$$\left| x - \frac{p}{q} \right| \leq \frac{k(q)}{q}$$

*holds infinitely often with Lebesgue measure 0 or 1, according as*

$$\sum_{n=1}^{\infty} k(n)$$

*is finite or not.*

Next, we define random variables that bridge the continued fraction digits  $(a_n)$  of an irrational number to its Diophantine properties (see e.g. [IK09, Chapter 1.2.1]).

**Lemma 1.11.** *Fix  $x := [a_0; a_1, \dots] \in \mathbb{R} \setminus \mathbb{Q}$ . Then with*

$$\begin{aligned} p_{-1} &:= 1, p_0 := a_0, q_{-1} := 0, q_0 := 1, \\ p_n &:= a_n p_{n-1} + p_{n-2}, q_n := a_n q_{n-1} + q_{n-2}, \\ r_n &:= \frac{1}{G^{n-1}} = [a_n; a_{n+1}, a_{n+2}, \dots]. \end{aligned}$$

*we have*

$$\begin{aligned} x &= \frac{p_{n-1} r_n + p_{n-2}}{q_{n-1} r_n + q_{n-2}}, \\ \frac{p_n}{q_n} &= a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}, \\ (-1)^k &= q_k p_{k-1} - p_k q_{k-1}. \end{aligned}$$

Setting

$$y_n := \frac{q_n}{q_{n-1}}, \tag{8}$$

$$u_n := q_{n-1}^{-2} \left| x - \frac{p_{n-1}}{q_{n-1}} \right|^{-1} \tag{9}$$

for  $n \in \mathbb{N}$ , we have  $q_n = y_1 \cdots y_n$  and  $y_n = [a_n; a_{n-1}, \dots, a_1] = a_n + y_{n-1}$ ,  $n \in \mathbb{N}$ . The random variable  $u_n$  is crucial in the context of diophantine approximations. Recall the well-known estimate

$$\frac{1}{q_{n-1} (q_n + q_{n-1})} < \left| x - \frac{p_{n-1}}{q_{n-1}} \right| < \frac{1}{q_{n-1} q_n}. \tag{10}$$

For a comprehensive account we refer to [DK02, Chapter 5] or [IK09].

As seen in the next lemma, the difference between the above defined variables and  $a_n$  is bounded.

**Lemma 1.12.** *Let  $(a_n)_n \in \mathbb{N}_0$  denote the digits of the continued fraction expansion and let the random variables  $r_n, y_n, u_n$ ,  $n \in \mathbb{N}_0$  be defined as above. Then it holds that*

- (A)  $a_n \leq r_n < a_n + 1$ ,  
(B)  $a_n \leq y_n < a_n + 1$ ,

(C)  $a_n < u_n < a_n + 2$ .

*Proof.* The inequalities (A) and (B) are immediate, (C) follows from Eq. (10).  $\square$

**Corollary 1.13.** *Let  $(c_n)_{n \in \mathbb{N}}$  be a sequence of real numbers and  $(d_n)_{n \in \mathbb{N}}$  be a sequence of natural numbers fulfilling the properties of Theorem 1.2 and such that  $\sum_{n=1}^{\infty} 1/(c_n d_n) = \infty$ . Then for the random variables,  $r_n, y_n$ , and  $u_n$ , associated to the continued fraction digits, as defined in Lemma 1.11 and Eq. (8) and (9), we have that the inequalities*

$$d_n < r_n, y_n \leq d_n (1 + 1/c_n) + 1$$

and

$$d_n < u_n \leq d_n (1 + 1/c_n) + 2$$

hold for infinitely many  $n \in \mathbb{N}$  Lebesgue almost everywhere.

*Proof.* Assume that  $\sum_{n=1}^{\infty} 1/(c_n d_n) = \infty$  holds. Then for  $(b_n) \in \{(r_n), (y_n)\}$ , applying (A) and (B) of Lemma 1.12, we can conclude with the above theorem that  $d_n < b_n \leq d_n (1 + 1/c_n) + 1$  infinitely often  $\lambda$ -almost everywhere (a.e.) and similarly, this time with (C) of Lemma 1.12, we get  $d_n < u_n \leq d_n (1 + 1/c_n) + 2$  infinitely often  $\lambda$ -a.e.  $\square$

## 2. MIXING PROPERTIES

Our results will depend crucially on the mixing properties of the continued fraction digits. For this we first introduce the classical notion of  $\phi$ - and  $\psi$ -mixing.

**Definition 2.1** Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space and  $\mathcal{C}, \mathcal{D} \subset \mathcal{A}$  two  $\sigma$ -fields, then the following quantity measures the dependence of the sub- $\sigma$ -fields.

$$\phi(\mathcal{C}, \mathcal{D}) := \sup_{\substack{C \in \mathcal{C}, D \in \mathcal{D} \\ \mathbb{P}(C) > 0}} |\mathbb{P}(D | C) - \mathbb{P}(D)|$$

$$\psi(\mathcal{C}, \mathcal{D}) := \sup_{\substack{C \in \mathcal{C}, D \in \mathcal{D} \\ \mathbb{P}(C), \mathbb{P}(D) > 0}} \left| \frac{\mathbb{P}(C \cap D)}{\mathbb{P}(C) \cdot \mathbb{P}(D)} - 1 \right|.$$

Let  $(X_n)_{n \in \mathbb{N}}$  be a (not necessarily stationary) sequence of random variables. For  $0 \leq J \leq L \leq \infty$  we can define a  $\sigma$ -field by

$$\mathcal{A}_J^L := \sigma(X_k, k \in \mathbb{N} \cap [J, L]).$$

With that the dependence coefficients are defined by

$$\phi(n) := \sup_{k \in \mathbb{N}} \phi(\mathcal{A}_0^k, \mathcal{A}_{k+n}^\infty),$$

$$\psi(n) := \sup_{k \in \mathbb{N}} \psi(\mathcal{A}_0^k, \mathcal{A}_{k+n}^\infty).$$

The sequence  $(X_n)$  is said to be  $\phi$ -mixing or  $\psi$ -mixing if  $\phi(n) \rightarrow 0$  or  $\psi(n) \rightarrow 0$  as  $n \rightarrow \infty$ .

It follows easily that

$$\phi(n) \leq \frac{1}{2} \psi(n), \tag{11}$$

for all  $n \in \mathbb{N}$ . For more details about mixing conditions see [Bra05].

Now we collect the necessary mixing properties of the continued fractions digits and state the following lemma from [IK09, Chapter 2.3.4].

**Lemma 2.2.** *Let  $\psi = \psi_{\mathfrak{m}}$  denote the  $\psi$ -mixing coefficient with respect to the continued fraction digits and the Gauss measure  $\mathfrak{m}$ . Then we have that*

$$\psi_{\mathfrak{m}}(n) \leq \rho \theta^{n-2} \text{ for } n \geq 2,$$

where  $\rho = \pi^2 \log 2 / 6 - 1$  and  $\theta$  is some constant less than 0.30367, and  $\psi_{\mathfrak{m}}(1) = 2 \log 2 - 1$ , i.e. the digits of the continued fraction expansion are exponentially  $\psi$ -mixing.

In the next lemma we state the more involved direction of the 0-1 law.

**Lemma 2.3.** Let  $(D_n)$  be a sequence of events such that  $D_n \in \sigma(a_n)$  for all  $n \in \mathbb{N}$ . If  $\sum_{n=1}^{\infty} \mathbf{m}(D_n) = \infty$ , then  $\mathbf{m}(\limsup_{n \rightarrow \infty} D_n) = 1$ .

To show this we make use of the mixing properties of the continued fractions using the following result of Chandra in [Cha08, Remark 1] for the reversed direction of the Borel-Cantelli Lemma.

**Lemma 2.4.** Let  $\mathbb{P}$  be a probability measure and  $(C_n)_{n \in \mathbb{N}}$  be a sequence of measurable subsets such that  $\sum_{n=1}^{\infty} \mathbb{P}(C_n) = \infty$ . If there exists a function  $q : \mathbb{N} \rightarrow \mathbb{R}_{>0}$  with  $\sum_{m=1}^{\infty} q(m) < \infty$  and such that for each  $i < j$  we have

$$\mathbb{P}(C_i \cap C_j) - \mathbb{P}(C_i)\mathbb{P}(C_j) \leq q(|i-j|) \cdot (\mathbb{P}(C_i) + \mathbb{P}(C_{i+1}) + \mathbb{P}(C_j) + \mathbb{P}(C_{j+1})),$$

then

$$\mathbb{P}\left(\limsup_{n \rightarrow \infty} C_n\right) = 1.$$

*Proof of Lemma 2.3.* By Lemma 2.2 and by  $\psi_{\mathbf{m}}$  defined therein we have that

$$\left| \frac{\mathbf{m}(D_i \cap D_j)}{\mathbf{m}(D_i) \cdot \mathbf{m}(D_j)} - 1 \right| \leq \psi_{\mathbf{m}}(|i-j|).$$

Thus,

$$\begin{aligned} |\mathbf{m}(D_i \cap D_j) - \mathbf{m}(D_i)\mathbf{m}(D_j)| &\leq \psi_{\mathbf{m}}(|i-j|) \cdot \mathbf{m}(D_i) \cdot \mathbf{m}(D_j) \\ &\leq \psi_{\mathbf{m}}(|i-j|) \cdot (\mathbf{m}(D_i) + \mathbf{m}(D_{i+1}) + \mathbf{m}(D_j) + \mathbf{m}(D_{j+1})). \end{aligned}$$

Clearly,  $\sum_{n=1}^{\infty} \psi(m) = \psi(1) + \sum_{m=0}^{\infty} \rho \theta^m < \infty$ . □

**Remark 2.5** The theorems from Section 1 use the  $\psi$ -mixing property of the continued fraction digits. The results could also be rephrased for a  $\phi$ -mixing sequence of random variables. Indeed, if  $(X_n)$  is a  $\phi$ -mixing sequence with summable  $\phi$ -mixing coefficients, then we have for  $i > j$  that

$$\begin{aligned} \mathbb{P}(C_i \cap C_j) - \mathbb{P}(C_i)\mathbb{P}(C_j) &\leq \phi(|i-j|) \cdot \mathbb{P}(C_j) \\ &\leq \phi(|i-j|) \cdot (\mathbb{P}(C_i) + \mathbb{P}(C_{i+1}) + \mathbb{P}(C_j) + \mathbb{P}(C_{j+1})). \end{aligned}$$

implying the mixing condition in Lemma 2.4 also for the  $\phi$ -mixing case.

We use Lemma 2.4 in particular to prove Theorems 1.2 and 1.5 and Corollary 1.3. For the proof of Theorem 1.6 we also use the mixing conditions in the proof of Lemma 5.2. In this case,  $\phi$ -mixing with a sufficiently small mixing coefficient is sufficient as well.

### 3. PROOFS OF THE 0-1 LAWS

*Proof of Theorem 1.2.* First we notice that (1) and (2) imply

$$\begin{aligned} \mathbf{m}\left(d_n \leq a_n \leq d_n + \frac{d_n}{c_n}\right) &= \frac{1}{\log 2} \cdot \left( \log\left(1 + \frac{1}{d_n}\right) - \log\left(1 + \frac{1}{d_n + \left\lfloor \frac{d_n}{c_n} \right\rfloor + 1}\right) \right) \\ &= \frac{1}{\log 2} \cdot \log\left( \frac{d_n + 1}{d_n} \cdot \frac{d_n + \left\lfloor \frac{d_n}{c_n} \right\rfloor + 1}{d_n + \left\lfloor \frac{d_n}{c_n} \right\rfloor + 2} \right). \end{aligned} \tag{12}$$

To prove the first part we assume that  $\sum_{n=1}^{\infty} (1/(c_n d_n) + 1/d_n^2) < \infty$  and notice that

$$\frac{d_n + \left\lfloor \frac{d_n}{c_n} \right\rfloor + 1}{d_n + \left\lfloor \frac{d_n}{c_n} \right\rfloor + 2} \leq \frac{d_n + \frac{d_n}{c_n} + 1}{d_n + \frac{d_n}{c_n} + 2}. \tag{13}$$

Thus,

$$\begin{aligned} \mathbf{m}\left(d_n \leq a_n \leq d_n + \frac{d_n}{c_n}\right) &\leq \frac{1}{\log 2} \cdot \log\left(1 + \frac{c_n + d_n}{c_n d_n^2 + 2c_n + d_n^2}\right) \\ &\leq \frac{1}{\log 2} \cdot \log\left(1 + \frac{1}{c_n d_n} + \frac{1}{d_n^2}\right). \end{aligned}$$

and hence

$$\log(2) \sum_{n \in \mathbb{N}} \mathbf{m} \left( d_n \leq a_n \leq d_n + \frac{d_n}{c_n} \right) \leq \sum_{n \in \mathbb{N}} \log \left( 1 + \frac{1}{c_n d_n} + \frac{1}{d_n^2} \right) \leq \sum_{n \in \mathbb{N}} \left( \frac{1}{c_n d_n} + \frac{1}{d_n^2} \right) < \infty.$$

By the Borel-Cantelli Lemma we conclude  $\mathbf{m}(d_n \leq a_n \leq d_n + d_n/c_n \text{ infinitely often}) = 0$ .

For proving the second part we first show that  $\sum_{n \in \mathbb{N}} 1/(c_n d_n) = \infty$  is a sufficient condition. To show this we make use of the mixing properties of the continued fractions using Lemma 2.3. Clearly,  $D_n := \{x : d_n \leq a_n(x) \leq d_n + d_n/c_n\} \in \sigma(a_n)$ . Thus,  $\sum_{n=1}^{\infty} \mathbf{m}(D_n) = \infty$  implies  $\mathbf{m}(\limsup_{n \rightarrow \infty} D_n) = 1$ . Next we show that  $\sum_{n \in \mathbb{N}} \mathbf{m}(D_n)$  diverges if  $\sum_{n \in \mathbb{N}} 1/(c_n d_n)$  does.

For this we come back to (12) and use this time

$$\frac{d_n + \left\lfloor \frac{d_n}{c_n} \right\rfloor + 1}{d_n + \left\lfloor \frac{d_n}{c_n} \right\rfloor + 2} > \frac{d_n + \frac{d_n}{c_n}}{d_n + \frac{d_n}{c_n} + 1}. \quad (14)$$

This together with (12) yields

$$\begin{aligned} \mathbf{m}(D_n) &\geq \frac{1}{\log 2} \cdot \log \left( 1 + \frac{1}{c_n d_n + 2c_n + d_n} \right) \\ &\geq \frac{1}{\log 2} \cdot \log \left( 1 + \frac{1}{4c_n d_n} \right). \end{aligned}$$

Hence, using  $x \log(2) \leq \log(1+x)$  for all  $x \in I$ , we get

$$\sum_{n \in \mathbb{N}} \mathbf{m}(D_n) \geq \frac{1}{\log 2} \sum_{n \in \mathbb{N}} \frac{\log 2}{4} \cdot \frac{1}{c_n d_n} = \infty.$$

Next we assume that  $\sum_{n \in \mathbb{N}} 1/d_n^2 = \infty$ . Clearly, for all  $n \in \mathbb{N}$ , we have

$$\{a_n = d_n\} \subset \left\{ d_n \leq a_n \leq d_n + \frac{d_n}{c_n} \right\}.$$

Since

$$\mathbf{m}(A_n) \geq \mathbf{m}(\{a_n = d_n\}) = \frac{1}{\log 2} \cdot \log \left( 1 + \frac{1}{d_n(d_n + 2)} \right)$$

and since  $x \log(2) \leq \log(1+x) \leq x$  for all  $x \in [0, 1]$ , we obtain

$$\sum_{n \in \mathbb{N}} \mathbf{m}(A_n) \geq \frac{1}{\log 2} \sum_{n \in \mathbb{N}} \frac{\log 2}{d_n(d_n + 2)} \geq \frac{1}{2} \sum_{n \in \mathbb{N}} \frac{1}{d_n^2} = \infty.$$

Using Lemma 2.3 we conclude that  $d_n \leq a_n \leq d_n(1 + 1/c_n)$  holds for infinitely many  $n \in \mathbb{N}$ ,  $\lambda$ -a.e. if  $\sum_{n \in \mathbb{N}} 1/(c_n d_n) = \infty$  or  $\sum_{n \in \mathbb{N}} 1/d_n^2 = \infty$ .  $\square$

*Proof of Theorem 1.5.* First we notice that (2) implies

$$\begin{aligned} \mathbf{m} \left( d_n < a_n \leq d_n + \frac{d_n}{c_n} \right) &= \frac{1}{\log 2} \cdot \left( \log \left( \frac{1}{d_n + 1} + 1 \right) - \log \left( \frac{1}{d_n + \left\lfloor \frac{d_n}{c_n} \right\rfloor + 1} + 1 \right) \right) \\ &= \frac{1}{\log 2} \cdot \log \left( \frac{d_n + 2}{d_n + 1} \cdot \frac{d_n + \left\lfloor \frac{d_n}{c_n} \right\rfloor + 1}{d_n + \left\lfloor \frac{d_n}{c_n} \right\rfloor + 2} \right). \end{aligned} \quad (15)$$

To prove the first part we assume that  $\sum_{n: c_n \leq d_n} 1/(c_n d_n) < \infty$ . Applying (13) on (15) yields

$$\mathbf{m} \left( d_n < a_n \leq d_n + \frac{d_n}{c_n} \right) \leq \frac{1}{\log 2} \cdot \log \left( 1 + \frac{1}{c_n d_n + 3c_n + d_n + 1 + 2 \cdot \frac{c_n}{d_n}} \right) \leq \frac{1}{\log 2} \cdot \log \left( 1 + \frac{1}{c_n d_n} \right).$$

Hence,

$$\log 2 \cdot \sum_{n: c_n \leq d_n} \mathbf{m} \left( d_n < a_n \leq d_n + \frac{d_n}{c_n} \right) \leq \sum_{n: c_n \leq d_n} \log \left( 1 + \frac{1}{c_n d_n} \right) \leq \sum_{n: c_n \leq d_n} \frac{1}{c_n d_n} < \infty.$$

By the Borel-Cantelli Lemma we conclude  $\mathbf{m}(d_n < a_n \leq d_n + d_n/c_n \text{ infinitely often}) = 0$ .

For proving the second part we make use of an analogous statement as in the proof of Theorem 1.2.

With  $A_n := \{x: d_n < a_n(x) \leq d_n + d_n/c_n\} \in \sigma(a_n)$  we have  $\mathbf{m}(A_n) > 0$  if and only if  $c_n \leq d_n$ .

Hence, we are left to show that  $\sum_{n \in \mathbb{N}} \mathbf{m}(A_n)$  diverges if  $\sum_{n: c_n \leq d_n} 1/(c_n d_n)$  does.

In the next steps let us assume that  $\lfloor d_n/c_n \rfloor = 1$ . Then we have that

$$\begin{aligned} \mathbf{m}\left(d_n < a_n \leq d_n + \frac{d_n}{c_n}\right) &= \frac{1}{\log 2} \cdot \log\left(1 + \frac{1}{d_n^2 + 4d_n + 3}\right) \\ &\geq \frac{1}{\log 2} \cdot \log\left(1 + \frac{1}{8d_n^2}\right) \\ &\geq \frac{1}{\log 2} \cdot \log\left(1 + \frac{1}{16c_n d_n}\right). \end{aligned}$$

In the next steps we assume that  $d_n/c_n \geq 2$ . Together with (14) and (15) this assumption yields

$$\begin{aligned} \mathbf{m}\left(d_n < a_n \leq d_n + \frac{d_n}{c_n}\right) &\geq \frac{1}{\log 2} \cdot \log\left(1 + \frac{1 - c_n/d_n}{c_n d_n + 2c_n + d_n + 1 + \frac{c_n}{d_n}}\right) \\ &\geq \frac{1}{\log 2} \cdot \log\left(1 + \frac{1 - 1/2}{c_n d_n + 2c_n + d_n + 1 + 1/2}\right) \\ &\geq \frac{1}{\log 2} \cdot \log\left(1 + \frac{1}{11c_n d_n}\right). \end{aligned}$$

Hence, using  $x \log 2 \leq \log(1+x)$  for all  $x \in [0, 1]$ , we get

$$\sum_{n: c_n \leq d_n} \mathbf{m}\left(d_n < a_n \leq d_n + \frac{d_n}{c_n}\right) \geq \frac{1}{\log 2} \sum_{n: c_n \leq d_n} \frac{\log 2}{11} \cdot \frac{1}{c_n d_n} = \infty.$$

Using Lemma 2.4 with  $\mathbb{P} := \mathbf{m}$  we conclude that  $d_n < a_n \leq d_n(1 + 1/c_n)$  holds for infinitely many  $n \in \mathbb{N}$ ,  $\lambda$ -a.e.  $\square$

#### 4. EXTENDED RANDOM VARIABLES

For the proof of the Central Limit Theorem we make use of the natural extension and an associated auxiliary family of measure to be introduced next.

To construct a doubly infinite version of  $(\bar{a}_n)_{n \in \mathbb{N}}$  under  $\mathbf{m}$  we use the natural extension. We first define  $\bar{G}: (0, 1) \times I \rightarrow (0, 1) \times I$  by

$$\bar{G}(\omega, \theta) := \left(G(\omega), \frac{1}{a_1(\omega) + \theta}\right).$$

It can be easily seen that

$$\bar{G}(\omega, \theta) = (G^n(\omega), [a_n(\omega), \dots, a_2(\omega), a_1(\omega) + \theta]).$$

Then we define  $(\bar{a}_k)_{k \in \mathbb{Z}}$ , where each  $\bar{a}_k: (0, 1) \times I \rightarrow \mathbb{N}$  by

$$\bar{a}_k(\omega, \theta) := \bar{a}_1(\bar{G}^k(\omega, \theta))$$

with

$$\bar{a}_1(\omega, \theta) := a_1(\omega).$$

Furthermore, we define the extended Gauss measure  $\bar{\mathbf{m}}$  on  $\mathcal{B}_{I^2}$  by

$$\bar{\mathbf{m}}(B) := \frac{1}{\log 2} \cdot \iint_B \frac{1}{(xy + 1)^2} dx dy$$

for  $B \in \mathcal{B}_{I^2}$ .  $\bar{G}$  preserves the extended Gauss measure  $\bar{\mathbf{m}}$ , see for example [IK09, Theorem 1.3.4]. In the following we give some important lemmas concerning the conditional distribution which are essential in the proof of Theorem 1.6.



**Lemma 4.1** (Theorem 1.3.5 of [IK09]). *For any  $x \in I$  we have for the conditional probability*

$$\overline{\mathbf{m}}([0, x] \times I : (\overline{a}_0, \overline{a}_{-1}, \dots)) = \frac{(a+1)x}{ax+1} \overline{\mathbf{m}}\text{-a.s.},$$

for the random variable  $a := [\overline{a}_0, \overline{a}_{-1}, \dots]$ .

Motivated by this lemma we also define the probability measure  $\mathbf{m}_a$  on  $\mathcal{B}_I$  via its distribution function, for  $a \in [0, 1]$ , by

$$\mathbf{m}_a([0, x]) := \frac{(a+1)x}{ax+1} \quad (16)$$

and have the following lemma.

**Lemma 4.2** (Brodén-Borel-Lévy formula, Proposition 1.3.8 of [IK09]). *For any  $a \in I$  we define  $s_0^a := a$  and  $s_n^a := 1/(s_{n-1}^a + a_n)$ . Then for any  $a \in I$  and  $n \in \mathbb{N}$  we have that*

$$\mathbf{m}_a(G^n < x : a_1, \dots, a_n) = \frac{(s_n^a + 1) \cdot x}{s_n^a x + 1}.$$

**Corollary 4.3** (Corollary 1.3.9 of [IK09]). *For any  $a, x \in I$  and  $n \in \mathbb{N}$  we have that*

$$\mathbf{m}_a(A|_{a_1, \dots, a_n}) = \mathbf{m}_{s_n^a}(G^n(A)),$$

for any  $A \in G^{-n}(\mathcal{B}_I)$ .

For further investigations of the extended version of  $(a_n)$  see [IK09, Section 1.3].

## 5. PROOF OF OF THE CENTRAL LIMIT THEOREMS

To prove Theorem 1.6 we start with two lemmas. In particular, Lemma 5.1 provides the exact value of  $\phi(1)$ . This improves a result by Philipp [Phi88, Lemma 2.1] who showed that  $\phi(1) < 0.4$ .

**Lemma 5.1.** *Let  $\phi = \phi_{\mathbf{m}}$  denote the  $\phi$ -mixing coefficient for the Gauss system. Then we have that*

$$\phi(1) = \frac{1 - \log 2 + \log \log 2}{\log 2} < 0.0861.$$

*Proof.* Let  $\mathbf{m}_a$  be the measure defined in (16) and let

$$\eta := \sup |\mathbf{m}_a(B) - \mathbf{m}(B)|$$

with the supremum taken over all  $a \in I$  and  $B \in \mathcal{B}_I$ . The proof of the lemma is separated into three parts, namely we show that

(A)  $\eta = (1 - \log 2 + \log \log 2) / \log 2$ ,

(B)  $\phi(1) \leq \eta$ , and

(C)  $\phi(1) \geq \eta$ .

The proofs of (B) and (C) are inspired by the proof for the  $\psi$ -mixing coefficient in [IK09].

ad (A): Let us define  $f : I^2 \rightarrow \mathbb{R}$  by

$$f(a, x) := \mathbf{m}_a([0, x]) - \mathbf{m}([0, x]) = \frac{(a+1)x}{ax+1} - \frac{\log(x+1)}{\log 2}.$$

We have that  $f(a, \cdot)$  is the distribution function of a signed measure with density  $\partial f(a, x) / \partial x$ . For each  $a \in I$  we obtain that  $\max_{\mathcal{B}_I} (\mathbf{m}_a(B) - \mathbf{m}(B))$  will be attained for  $B = \{x : \partial f(a, x) / \partial x > 0\}$  and  $\min \mathbf{m}_a(B) - \mathbf{m}(B)$  will be attained for  $B^c$ . In the following we will only calculate  $\min \mathbf{m}_a(B) - \mathbf{m}(B)$  since

$$\mathbf{m}_a(B^c) - \mathbf{m}(B^c) = 1 - \mathbf{m}_a(B) - (1 - \mathbf{m}(B)) = -(\mathbf{m}_a(B) - \mathbf{m}(B))$$

and thus  $\max_{a \in I} \mathbf{m}_a(B) - \mathbf{m}(B) = -\min_{a \in I} \mathbf{m}_a(B) - \mathbf{m}(B)$

In the next steps we calculate the zeros of  $\partial f(a, x) / \partial x$  in dependency of  $a$ . We have that

$$\frac{\partial f(a, x)}{\partial x} = \frac{a+1}{(ax+1)^2} - \frac{1}{\log 2 \cdot (x+1)}.$$

From this we find that the two zeros are given by

$$x_{a,1} = \frac{(a+1) \cdot \log 2 - 2a}{2a^2} + \sqrt{\left(\frac{(a+1) \cdot \log 2 - 2a}{2a^2}\right)^2 - \frac{1 - (a+1) \cdot \log 2}{a^2}} \text{ and}$$

$$x_{a,2} = \frac{(a+1) \cdot \log 2 - 2a}{2a^2} - \sqrt{\left(\frac{(a+1) \cdot \log 2 - 2a}{2a^2}\right)^2 - \frac{1 - (a+1) \cdot \log 2}{a^2}}.$$

With some further analysis we obtain that  $x_{a,1} \in I$  if and only if  $a \in [2 \log 2 - 1, 1]$  and  $x_{a,2} \in I$  if and only if  $a \in [0, 1/\log 2 - 1]$ .

We have that  $\partial f(a, x)/\partial x$  changes sign from plus to minus in  $x = x_{a,1}$  for  $a \in [2 \log 2 - 1, 1]$  and  $\partial f(a, x)/\partial x$  changes sign from minus to plus in  $x = x_{a,2}$  for  $a \in [0, 1/\log 2 - 1]$ .

We consider in the following three cases, namely

- (a)  $0 \leq a < 2 \log 2 - 1$ ,
- (b)  $2 \log 2 - 1 \leq a \leq 1/\log 2 - 1$ , and
- (c)  $1/\log 2 - 1 < a \leq 1$ .

ad (a): In this case we have that  $\min_{B \in \mathcal{B}_I} \mathbf{m}_a(B) - \mathbf{m}(B)$  will be attained for  $B = [0, x_{a,2}]$ .

By determining the partial derivative of  $f$  with respect to  $a$

$$\frac{\partial f(a, x)}{\partial a} = \frac{x \cdot (1 - x)}{(ax + 1)^2}$$

we obtain that for all  $x \in I$  we have that  $\partial f(a, x)/\partial a \geq 0$ , i.e. for all  $x \in I$ ,  $f$  is monotonically increasing in  $a$ . Thus,  $\min_{a \in [0, 2 \log 2 - 1], B \in \mathcal{B}_I} \mathbf{m}_a(B) - \mathbf{m}(B) = \min_{B \in \mathcal{B}_I} \mathbf{m}_0(B) - \mathbf{m}(B)$ . Using  $x_{0,2} = 1/\log 2 - 1$  we find

$$f\left(0, \frac{1}{\log 2} - 1\right) = \frac{1 - \log 2 + \log \log 2}{\log 2}$$

and consequently

$$\min_{a \in [0, 2 \log 2 - 1], B \in \mathcal{B}_I} \mathbf{m}_a(B) - \mathbf{m}(B) = \frac{1 - \log 2 + \log \log 2}{\log 2}. \quad (17)$$

ad (b): In this case we have that  $\min_{B \in \mathcal{B}_I} (\mathbf{m}_a(B) - \mathbf{m}(B))$  will be attained for  $B = [0, x_{a,2}) \cap [x_{a,1}, 1]$ . Furthermore, by the monotonicity of  $f$  in  $a$  we have that

$$\begin{aligned} & \min_{a \in [2 \log 2 - 1, 1/\log 2 - 1], B \in \mathcal{B}_I} \mathbf{m}_a(B) - \mathbf{m}(B) \\ & \geq \min_{a \in [2 \log 2 - 1, 1/\log 2 - 1]} \mathbf{m}_a([0, x_{a,2})) - \mathbf{m}([0, x_{a,2})) \end{aligned} \quad (18)$$

$$+ \min_{b \in [2 \log 2 - 1, 1/\log 2 - 1]} \mathbf{m}_b([x_{b,1}, 1]) - \mathbf{m}([x_{b,1}, 1]). \quad (19)$$

For (18) we have by the monotonicity of  $f$  in  $a$  that the minimum will be attained for  $a = 2 \log 2 - 1$  in  $x_{2 \log 2 - 1, 2}$  and for (19) we again have by the monotonicity of  $f$  in  $a$  that the minimum will be attained for  $a = 1/\log 2 - 1$  in  $x_{1/\log 2 - 1, 1}$ .

Thus, on the one hand,

$$\min_{a \in [2 \log 2 - 1, 1/\log 2 - 1]} \mathbf{m}_a([0, x_{a,2})) - \mathbf{m}([0, x_{a,2})) = f(2 \log 2 - 1, x_{2 \log 2 - 1, 2}) \quad (20)$$

On the other hand, we have

$$\begin{aligned} \min_{b \in [2 \log 2 - 1, 1/\log 2 - 1]} \mathbf{m}_b([x_{b,1}, 1]) - \mathbf{m}([x_{b,1}, 1]) &= -f(1/\log 2 - 1, x_{1/\log 2 - 1, 1}) \\ &= f(2 \log 2 - 1, x_{2 \log 2 - 1, 2}). \end{aligned} \quad (21)$$

Combining (18) and (19) with (20) and (21) yields

$$\min_{a \in [2 \log 2 - 1, 1/\log 2 - 1], B \in \mathcal{B}_I} \mathbf{m}_a(B) - \mathbf{m}(B) \geq 2 \cdot f(2 \log 2 - 1, x_{2 \log 2 - 1, 2}) \geq -0.0118 \quad (22)$$

*ad (c):* In this case we have that  $\min_{B \in \mathcal{B}_I} (\mathfrak{m}_a(B) - \mathfrak{m}(B))$  will be attained for  $B = [x_{a,1}, 1]$ . Furthermore, since  $f(a, 1) = 0$ , and by the continuity of  $f$  we have that

$$\begin{aligned} \min_{a \in (1/\log 2 - 1, 1]} \mathfrak{m}_a([x_{a,1}, 1]) - \mathfrak{m}([x_{a,1}, 1]) &= \min_{a \in (1/\log 2 - 1, a]} -f(a, x_{a,1}) \\ &= -f(1, x_{1,1}) \\ &= -f(1, 2\log 2 - 1) \\ &= \frac{1 - \log 2 + \log \log 2}{\log 2}. \end{aligned} \quad (23)$$

Putting (17), (22), and (23) together yields the first statement.

*ad (B):* We define for  $n \in \mathbb{N}$  and  $i^{(n)} \in \mathbb{N}^n$

$$I(i^{(n)}) := \{a_k = i_k, 1 \leq k \leq n\}$$

and obtain from Lemma 4.3 and the fact that  $G$  is  $\mathfrak{m}$ -invariant that

$$\eta = \sup \left| \mathfrak{m}_a(B|i^{(k)}) - \mathfrak{m}(B) \right|,$$

where the supremum is taken over all  $B \in \mathcal{B}_{k+n}^\infty$  with  $\mathfrak{m}(B) > 0$ ,  $i^{(k)} \in \mathbb{N}^k$ , and  $k \in \mathbb{N}$ .

Thus, if we set  $A := I(i^{(k)})$ ,  $B := G^{-k}I(j^{(m)})$  for arbitrarily given  $k, m, n \in \mathbb{N}$ ,  $i^{(k)} \in \mathbb{N}^k$ , and  $j^{(m)} \in \mathbb{N}^m$ , we obtain that

$$|\mathfrak{m}_a(B|A) - \mathfrak{m}(B)| \leq \eta$$

and thus

$$|\mathfrak{m}_a(B \cap A) - \mathfrak{m}(B) \cdot \mathfrak{m}(A)| \leq \eta \cdot \mathfrak{m}(A).$$

By integrating the above inequality over  $a \in I$  with respect to  $\mathfrak{m}$  and considering that

$$\int_a \mathfrak{m}_a(E) da = \mathfrak{m}(E)$$

we obtain  $\phi(1) \leq \eta$ .

*ad (C):* To prove the third part of the lemma we make use of the extended version of the Gauss system. We have that

$$\phi(1) = \sup |\overline{\mathfrak{m}}(\overline{A}|\overline{B}) - \overline{\mathfrak{m}}(\overline{A})|,$$

where the supremum is taken over  $\overline{A} \in \sigma(\overline{a}_n, \overline{a}_{n+1}, \dots)$  and  $\overline{B} \in \sigma(\overline{a}_0, \overline{a}_{-1}, \dots)$  for which  $\overline{\mathfrak{m}}(\overline{B}) > 0$ . This follows directly from the definition of the biinfinite sequence  $(\overline{a}_n)_{n \in \mathbb{Z}}$  and the definition of the  $\phi$ -mixing coefficient.

Clearly,  $\overline{A} = A \times I$  and  $\overline{B} = I \times B$ , with  $A \in \mathcal{B}_I^\infty := \mathcal{B}_I$  and  $B \in \mathcal{B}_I$ . Thus,

$$\phi(1) = \sup \left| \frac{\overline{\mathfrak{m}}(A \times B)}{\mathfrak{m}(A)} - \mathfrak{m}(B) \right| \quad (24)$$

with the supremum taken over  $A, B \in \mathcal{B}_I$  and  $\mathfrak{m}(A) > 0$ . Furthermore, we have that

$$\overline{\mathfrak{m}}(A \times B) = \int_A \mathfrak{m}(da) \mathfrak{m}_a(B) = \int_B \mathfrak{m}(db) \mathfrak{m}_b(A)$$

for any  $A, B \in \mathcal{B}_I$ . It follows from (24) and the definition of  $\eta$  that

$$\phi(1) \leq \sup_{a \in I, B \in \mathcal{B}_I} |\mathfrak{m}_a(B) - \mathfrak{m}(B)| = \eta,$$

which completes the proof of (C). □

**Lemma 5.2.** *If (4) holds, then  $\lim_{n \rightarrow \infty} \mathbb{V}(\sum_{i=1}^n \mathbb{1}_{A_i}) = \infty$ .*

*Proof.* We first notice that by Lemma 2.3 we have that (4) implies  $\sum_{n=1}^\infty \mathfrak{m}(B_n) = \infty$ .

Since  $\{a_n = 1\} \subset A_n \subsetneq \{a_n \in \mathbb{N}\}$  for only finitely many  $n \in \mathbb{N}$ , say for no  $n$  greater than  $N \in \mathbb{N}$ . Without loss of generality we assume that  $N = 1$ . We have that

$$\begin{aligned} \mathbb{V}(S_n) &= \mathbb{V}\left(\sum_{k=1}^n \mathbb{1}_{A_k}\right) \\ &= \sum_{i=1}^n \left( \mathbb{V}(\mathbb{1}_{A_i}) + 2 \sum_{j=1}^{i-1} \text{Cov}(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}) \right). \end{aligned} \quad (25)$$

Estimating the inner summands in (25) we first notice that

$$\mathbb{V}(\mathbb{1}_{\{a_i \in \mathbb{N}\}}) + 2 \sum_{j=1}^{i-1} \text{Cov}(\mathbb{1}_{\{a_i \in \mathbb{N}\}}, \mathbb{1}_{A_j}) = 0.$$

Assume now that  $\{a_i = 1\} \cap A_i = \emptyset$ . We always have that

$$\mathbb{V}(\mathbb{1}_{A_i}) = \mathfrak{m}(A_i) \cdot \mathfrak{m}(A_i^c)$$

and for  $i > j$  we have that

$$\text{Cov}(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}) = \mathfrak{m}(A_i | A_j) - \mathfrak{m}(A_i) \geq -\phi(i-j) \cdot \mathfrak{m}(A_i).$$

Thus,

$$\begin{aligned} \mathbb{V}(\mathbb{1}_{A_i}) + 2 \sum_{j=i+1}^n \text{Cov}(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}) &\geq \mathfrak{m}(A_i) \cdot \mathfrak{m}(A_i^c) - 2 \sum_{j=1}^{i-1} \phi(i-j) \cdot \mathfrak{m}(A_i) \\ &= \mathfrak{m}(A_i) \cdot \left( \mathfrak{m}(A_i^c) - 2 \sum_{j=1}^{i-1} \phi(i-j) \right) \\ &\geq \mathfrak{m}(A_i) \cdot \left( (1 - \rho + \epsilon) - 2 \sum_{j=1}^{i-1} \phi(i-j) \right) \\ &\geq \mathfrak{m}(A_i) \cdot \left( (1 - \rho + \epsilon) + 2\phi(1) - \sum_{j=2}^{i-1} \psi(j) \right), \end{aligned}$$

which follows from (11).

Using the estimates of Lemma 2.2 and Lemma 5.1 we have that

$$\begin{aligned} \mathbb{V}(\mathbb{1}_{A_i}) + 2 \sum_{j=i+1}^n \text{Cov}(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}) &\geq \mathfrak{m}(A_i) \cdot \left( (1 - \rho + \epsilon) - \frac{1 - \log 2 + \log \log 2}{\log 2} - 2 \left( \frac{\pi^2 \cdot \log 2}{6} - 1 \right) \cdot \sum_{j=0}^{\infty} \theta^j \right) \\ &= \mathfrak{m}(A_i) \cdot ((1 - \rho + \epsilon) - (1 - \rho)) \\ &= \mathfrak{m}(A_i) \cdot \epsilon. \end{aligned}$$

Hence, we have for the sum in (25) that  $\mathbb{V}(S_n) \geq \epsilon \cdot \sum_{i=1}^n \mathfrak{m}(B_i)$ . Thus, if  $\sum_{n=1}^{\infty} \mathfrak{m}(B_n) = \infty$ , then  $\lim_{n \rightarrow \infty} \mathbb{V}(S_n) = \infty$ . □

*Proof of Theorem 1.6.* We use a Theorem by Neumann, see [Neu10, Theorem 2.1]:

**Lemma 5.3.** *Let  $(X_{n,k})_{n \in \mathbb{N}, k \leq n}$  be a triangular array of random variables with zero expectation such that there exists  $\nu_0$  with  $\sum_{k=1}^n \mathbb{E}(X_{n,k}^2) < \nu_0$  for all  $n \in \mathbb{N}$ . Furthermore, we assume that*

$$\lim_{n \rightarrow \infty} \mathbb{V}\left(\sum_{i=1}^n X_{n,i}\right) =: \sigma^2 \in [0, \infty) \quad (26)$$

and for all  $\epsilon > 0$  it holds that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E} (X_{n,i}^2 \cdot \mathbb{1}_{\{|X_{n,i}| > \epsilon\}}) = 0.$$

Moreover, we assume that there exists a summable sequence  $(\theta_r)_{r \in \mathbb{N}}$  such that, for all  $u \in \mathbb{N}$  and all indices  $1 \leq s_1 < s_2 < \dots < s_u < s_u + r = t_1 \leq t_2 \leq n$ , the following upper bounds for covariances hold true:

(A) for all measurable and quadratic integrable functions  $f : \mathbb{R}^u \rightarrow \mathbb{R}$  holds that

$$|\text{Cov}(f(X_{n,s_1}, \dots, X_{n,s_u}), X_{n,t_1})| \leq \theta_r \cdot \sqrt{\mathbb{E}(f^2(X_{n,s_1}, \dots, X_{n,s_u}))} \cdot \max \left\{ \sqrt{\mathbb{E}(X_{n,t_1}^2)}, n^{-1/2} \right\}, \quad (27)$$

(B) for all measurable and bounded functions  $f : \mathbb{R}^u \rightarrow \mathbb{R}$  holds that

$$|\text{Cov}(f(X_{n,s_1}, \dots, X_{n,s_u}), X_{n,t_1} \cdot X_{n,t_2})| \leq \theta_r \cdot |f|_\infty \cdot (\mathbb{E}(X_{n,t_1}^2) + \mathbb{E}(X_{n,t_2}^2) + n^{-1}). \quad (28)$$

Then

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n X_{n,k} = \mathcal{N}(0, \sigma^2)$$

in distribution.

We apply this lemma to the random variables

$$(X_{n,k}) := \left( \frac{\mathbb{1}_{\{a_k \geq c_k\}} - \mathbf{m}(a_k \geq c_k)}{\mathbb{V}(\sum_{i=1}^n \mathbb{1}_{\{a_i \geq c_i\}})} \right),$$

which are  $\psi$ -mixing since the continued fraction digits are  $\psi$ -mixing by Lemma 2.2. Since the  $(X_{n,k})$  are centered, (26) follows immediately.

Since  $\lim_{n \rightarrow \infty} \mathbb{V}(\sum_{i=1}^n \mathbb{1}_{A_i}) = \infty$  by Lemma 5.2 we obtain that for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$  we have that  $\mathbb{1}_{\{|X_{n,k}| > \epsilon\}} = 0$  and thus (26) follows.

To prove (A) we use [Bil68, Lemma 20.1] which states the following.

**Lemma 5.4.** *Let  $(Y_n)_{n \in \mathbb{N}}$  be a stationary,  $\phi$ -mixing process. For  $a \leq b$  let  $\mathcal{M}_a^b$  be the  $\sigma$ -field generated by the random variables  $Y_a, \dots, Y_b$  and  $\mathcal{M}_a^\infty$  the  $\sigma$ -field generated by the random variables  $Y_a, Y_{a+1}, \dots$ . Let  $g$  be  $\mathcal{M}_0^k$ -measurable and  $h$  be  $\mathcal{M}_{k+n}^\infty$ -measurable with  $|h|_\infty < \infty$ . Furthermore, let us assume that for  $r, s > 1$  with  $1/r + 1/s = 1$  we have  $\mathbb{E}(|g|^r) < \infty$  and  $\mathbb{E}(|h|^s) < \infty$ . Then*

$$|\mathbb{E}(gh) - \mathbb{E}(g)\mathbb{E}(h)| \leq 2\phi(n)^{1/r} \cdot \mathbb{E}(|g|^r)^{1/r} \cdot \mathbb{E}(|h|^s)^{1/s}.$$

Setting  $r := s := 2$ ,  $g := f(X_{n,s_1}, \dots, X_{n,s_u})$ , and  $h := X_{n,t_1}$  a straight forward application of Lemma 5.4 yields

$$\begin{aligned} |\text{Cov}(f(X_{n,s_1}, \dots, X_{n,s_u}), X_{n,t_1})| &\leq 2\phi(n)^{1/2} \cdot \sqrt{\mathbb{E}(f^2(X_{n,s_1}, \dots, X_{n,s_u}))} \cdot \sqrt{\mathbb{E}(X_{n,t_1}^2)} \\ &\leq 2\phi(n)^{1/2} \cdot \sqrt{\mathbb{E}(f^2(X_{n,s_1}, \dots, X_{n,s_u}))} \cdot \max \left\{ \sqrt{\mathbb{E}(X_{n,t_1}^2)}, n^{-1/2} \right\}. \end{aligned}$$

Furthermore, to prove (28) we use the following Lemma which is [Bil68, (20.28)].

**Lemma 5.5.** *Let the assumption be as in Lemma 5.4. Then*

$$|\mathbb{E}(gh) - \mathbb{E}(g)\mathbb{E}(h)| \leq 2\phi(n) \cdot \mathbb{E}(|g|) \cdot |h|_\infty.$$

Now, this lemma gives with  $g := X_{n,t_1} \cdot X_{n,t_2}$  and  $h := f(X_{n,s_1}, \dots, X_{n,s_u})$  that

$$\begin{aligned} |\text{Cov}(f(X_{n,s_1}, \dots, X_{n,s_u}), X_{n,t_1} \cdot X_{n,t_2})| &\leq 2\phi(n) \cdot |f|_\infty \cdot \mathbb{E}(|X_{n,t_1} \cdot X_{n,t_2}|) \\ &\leq 2\phi(n) \cdot |f|_\infty \cdot \mathbb{E}(\max\{X_{n,t_1}^2, X_{n,t_2}^2\}) \\ &\leq 2\phi(n)^{1/2} \cdot |f|_\infty \cdot (\mathbb{E}(X_{n,t_1}^2) + \mathbb{E}(X_{n,t_2}^2) + n^{-1}). \end{aligned}$$

We have that  $\phi(n) \leq \psi(n)/2$ , see for example [Bra05, (1.11)], and by Lemma 2.2 it follows that  $\sum_{n=1}^\infty 2\phi(n)^{1/2} < \infty$ . Setting  $\theta_r := 2\phi(r)^{1/2}$ , (27) and (28) follow.

Hence, an application of Lemma 5.3 proves our central limit theorem.  $\square$

*Proof of Corollary 1.9.*  $ad(A)$ : We have that

$$B_n = \begin{cases} A_n & \text{if } b_n > 1 \\ 0 & \text{else.} \end{cases}$$

Thus,

$$\sum_{n=1}^{\infty} \mathbf{m}(\mathbb{1}_{B_n}) = \sum_{b_n > 1} \frac{1}{\log 2} \log \left( 1 + \frac{1}{\lceil b_n \rceil} \right). \quad (29)$$

Since  $x \cdot \log(2) \leq \log(1+x) \leq x$  for all  $x \in [0, 1]$  it follows that  $\sum_{b_n > 1} 1/b_n = \infty$  is equivalent to (29) tending to infinity. Thus, if  $\sum_{b_n > 1} 1/b_n = \infty$ , then  $\sum_{n=1}^{\infty} \mathbf{m}(\mathbb{1}_{B_n}) = \infty$  and by Lemma 2.3 we can conclude that (4) holds. Further, by Remark 1.7 we have that  $\mathbf{m}(B_n) < \rho$  for all  $n \in \mathbb{N}$  and thus, the assumptions in Theorem 1.6 are fulfilled.

$ad(B)$ : (4) follows immediately since this is a special case of the assumptions in Theorem 1.2. Since  $(d_n)$  tends to infinity, we have that  $A_n \subset \{a_n > 1\}$  for all  $n$  sufficiently large. Thus, by Remark 1.7 we have that  $\mathbf{m}(B_n) < \rho$  for all  $n$  sufficiently large.

$ad(C)$ : We have that  $B_n = A_n$  for  $n$  sufficiently large and obtain from the proof of Theorem 1.2 that (4) holds if  $\sum_{n \in \mathbb{N}} 1/(c_n d_n) = \infty$  or  $\sum_{n \in \mathbb{N}} 1/d_n^2 = \infty$ . Since  $(d_n)$  tends to infinity, we have that  $A_n \subset \{a_n > 1\}$  for all  $n$  sufficiently large. Thus, by Remark 1.7 we have that  $\mathbf{m}(B_n) < \rho$  for all  $n$  sufficiently large.

$ad(D)$ : We have that  $B_n = A_n$  for  $n$  sufficiently large and obtain from the proof of Theorem 1.5 that (4) holds if  $\sum_{n: c_n \leq d_n} 1/(c_n d_n) = \infty$ . Since  $(d_n)$  tends to infinity, we have that  $A_n \subset \{a_n > 1\}$  for all  $n$  sufficiently large. Thus, by Remark 1.7 we have that  $\mathbf{m}(B_n) < \rho$  for all  $n$  sufficiently large. □

## REFERENCES

- [Ber12a] F. Bernstein. Über geometrische Wahrscheinlichkeit und über das Axiom der beschränkten Arithmetisierbarkeit der Beobachtungen. *Math. Ann.*, 72(4):585–587, 1912.
- [Ber12b] F. Bernstein. Über eine Anwendung der Mengenlehre auf ein aus der Theorie der säkularen Störungen herrührendes Problem. *Math. Ann.*, 71:417–439, 1912.
- [Bil68] P. Billingsley. *Convergence of probability measures*. John Wiley & Sons, Inc., New York-London-Sydney, 1968.
- [Bor09] E. Borel. Les probabilités denombrables et leurs applications arithmétiques. *Rend. Circ. Mat. Palermo*, 27:247–271, 1909.
- [Bra05] R. C. Bradley. Basic properties of strong mixing conditions. A survey and some open questions. *Probability Surveys*, 2:107–144, 2005.
- [Cha08] T. K. Chandra. The Borel-Cantelli lemma under dependence conditions. *Statistics & Probability Letters*, 78(4):390–395, 2008.
- [DK02] K. Dajani and C. Kraaikamp. *Ergodic Theory of Numbers*. The Mathematical Association of America, July 2002.
- [Gal72] J. Galambos. Some remarks on the Lüroth expansion. *Czechoslovak Mathematical Journal*, 22(2):266–271, 1972.
- [IK09] M. Iosifescu and C. Kraaikamp. *Metrical Theory of Continued Fractions*. Springer Netherlands, 2002. edition, December 2009.
- [Khi35] A. Y. Khintchine. Metrische Kettenbruchprobleme. *Compositio Math*, 1:361–382, 1935.
- [Neu10] M. H. Neumann. A central limit theorem for triangular arrays of weakly dependent random variables. *Technical report on Stochastics and Statistics, Faculty of Mathematics and Computer Sciences of the Friedrich-Schiller-Universität Jena*, 2010.
- [Phi88] W. Philipp. Limit theorems for sums of partial quotients of continued fractions. *Monatshefte für Mathematik*, 105(3):195–206, 1988.

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